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CONVERGENCE ANALYSIS OF QUANTUM-INSPIRED EVOLUTIONARY ALGORITHMS BASED ON THE BANACH FIXED POINT THEOREM

ROBERT NOWOTNIAK, JACEK KUCHARSKI

Computer Engineering Department
Technical University of Lodz, 90-924 Lodz, Stefanowskiego 18/22, Poland
e-mail: {rnowotniak, jkuchars}@kis.p.lodz.pl

In this paper a convergence analysis of general Quantum-Inspired Evolutionary Algorithm based on the Banach fixed point theorem has been performed. A new quality measure for quantum genotypes has been proposed and the algorithm has been considered in the relevant metric space. Necessary conditions for the convergence of the algorithm have been discussed. Finally, the analysis has been illustrated by a Quantum-Inspired Genetic Algorithm.

Keywords: quantum-inspired genetic algorithms, evolutionary computing, quantum computing, convergence analysis, optimization

1. Introduction

This paper considers the convergence of evolutionary algorithms, drawing their inspiration from both biological evolution and unitary evolution of quantum systems (Chuang and Nielsen, 2000). The analysis has been performed for a general *Quantum-Inspired Evolutionary Algorithm* – an iterative optimization algorithm, exploiting concepts and principles of quantum computation in selected stages (representation of solutions, genetic operators, evaluation of individuals' fitness). Our analysis has been illustrated by *Quantum-Inspired Genetic Algorithm* proposed in (Han and Kim, 2000).

Several different approaches are possible to theoretical analysis of performance and convergence of evolutionary algorithms: Holland's schema theorem (Holland, 1975), Banach fixed point theorem (Michalewicz and Szałas, 1993; Michalewicz, 1996), finite Markov chains (Goldberg and Segrest, 1987) and Walsh transforms (Goldberg, 1989). Each of these methods allows to identify certain properties of evolutionary algorithms and their limitations in different areas. For example, the Banach fixed point theorem allows to prove the convergence of selected evolutionary algorithms, while Holland's schema theorem explains efficiency of the algorithms by propagation of over-averaged schemas in consecutive iterations. All these methods identify some lower and upper boundaries of evolutionary algorithms performance.

To perform the analysis based on the Banach fixed point theorem, we will consider the algorithm in a metric population space. Consecutive iterations of the algorithm in the selected space are contraction mapping of quantum populations. As by the Banach fixed point theorem every such contraction has unique fixed point, the convergence of considered algorithm will be proved by this line of reasoning.

This paper is organised as follows. In section 2 fundamentals of quantum-inspired evolutionary algorithms have been described. In section 3 the Banach fixed point theorem has been formulated and related definitions have been provided. The main contribution of this paper, the convergence analysis of Quantum-Inspired Evolutionary Algorithm, has been performed in section 4. In section 5 demonstration of convergence analysis for an exemplary quantum genetic algorithm has been presented. Section 6 draws final conclusions and summarizes main points of this paper.

2. Quantum-Inspired Evolutionary Algorithms

The first proposal of evolutionary algorithm based on the concepts and principles of quantum computing was presented in (Narayanan and Moore, 1996) and this area is still intensively studied nowadays. However, the theoretical aspect of the new algorithms has not been thoroughly studied with all due attention in the past decade.

The *quantum-inspired evolutionary algorithms* are located in the intersection of two subfields of computer science: quantum computing and evolutionary computing. The considered algorithms do not require, however, a functional quantum computer for their efficient implementation, but they exploit additional level of randomness inspired by quantum mechanical systems.

To simplify the notation, the algorithms will be denoted henceforth in this paper as *quantum evolutionary algorithms*, thought no real quantum-level hardware is required for their implementation. However, the possibility of potential implementation of the algorithms on a quantum computer is quite probable in the future.

The significant feature of the new algorithms is the representation of solutions. Contrary to traditional evolutionary algorithms, each quantum population consists of probability distributions of sampling the search space, instead of exact points in this space. In the special case, however, a quantum individual can indicate an exact element in the space.

Let us denote by X the search space, by Ω – universal sample space consisting of possible outcomes of sampling X and by $\mathcal{F}_\Omega$ – σ-algebra over the sample space Ω. Fundamental terms of quantum evolutionary algorithms are described below:

- *quantum gene* g – a single element of quantum chromosome. It encodes a probability distribution $Pr_g : \mathcal{F}_G \mapsto [0, 1]$ of observing the gene values from the set G.

  For example, in the case of binary quantum representation ($G = \{0, 1\}$), such *binary quantum gene* g, encoding probability distribution $Pr_g : \mathcal{F}_{\{0,1\}} \mapsto [0, 1]$, can be represented as a qubit, with its imaginary part possibly omitted.

- *binary quantum chromosome* $g_1 g_2 \dots g_m$ – a string of m binary quantum genes, each coding probabilities of two possible outcomes. Thus, the binary quantum chromosome corresponds to a Cartesian product of m probability distributions $Pr_{g1} \times Pr_{g2} \times \dots \times Pr_{gm}$.

- *quantum genotype* q – a set of quantum chromosomes (in the special case, containing only one element, $q = \{g_1 g_2 \dots g_m\}$) or other data structure, encoding probabilities of individual's features expression. Quantum genotype q encodes probability distribution $Pr_q : \mathcal{F}_\Omega \mapsto [0, 1]$ of sampling the search space X by the algorithm.

- *quantum population* $Q = \{q_1, q_2, \dots, q_k\}$ – a set of quantum genotypes. Therefore, quantum population Q is isomorphic to a set of k probability spaces $\{(\Omega, \mathcal{F}_\Omega, Pr_1), (\Omega, \mathcal{F}_\Omega, Pr_2), \dots, (\Omega, \mathcal{F}_\Omega, Pr_k)\}$.

The pseudocode of general Quantum-Inspired Evolutionary Algorithm has been presented in Fig. 2.

procedure QIEA
begin
$t \leftarrow 0$
initialize $Q(0)$
...
while not termination-criterion **do**
  $t \leftarrow t + 1$
  ...
  evaluate $Q(t)$
  perform *genetic operators* on $Q(t)$
  ...
end while
end

Fig. 1. General Quantum-Inspired Evolutionary Algorithm

It is easy to notice that the pseudocode above corresponds directly to the general classical evolutionary algorithm. However, the three emphasized stages of the quantum evolutionary algorithm are modelled upon concepts and principles of quantum computing:

Initialization – Initial quantum population $Q(0)$ does not have to specify exact (randomly chosen) elements of the search space X. Instead, $Q(0)$ can be initialized by some probability distributions $Pr_{1,\dots,k}$ of sampling the space. Uniform distributions or other families of distributions (e.g. normal or triangle distributions) with random parameters can be used.

Quantum genetic operators – genetic operators in quantum evolutionary algorithms are adapted to the new representation. Two possible groups of the new genetic operators can be identified: generalization of classical genetic operators to the new quantum representation and a new class of operators, modelling directly quantum mechanical phenomena (e.g. rotations of quantum systems state vectors).

Evaluation of individuals' quality – quantum individuals are chosen for reproduction according to their fitness. Various stochastic and deterministic measures of quality for quantum genotypes are possible at this stage of the algorithm.

3. Banach Fixed Point Theorem

This section provides the Banach Fixed point theorem and definitions required for the proof of convergence.

Definition 1.

Let S be a non-empty set. The *metric* on S is a function $\delta : S \times S \mapsto \mathbb{R}$ such that for any $a, b, c \in S$:

- $\delta(a, b) = 0 \iff a = b$ (identity of indiscernibles)
- $\delta(a, b) = \delta(b, a)$ (symmetry)

- $\delta(a,b) \leq \delta(a,c) + \delta(c,b)$ (triangle inequality)

The *metric space* is an ordered pair $\langle S, \delta \rangle$ where S is a non-empty set and δ is a metric on S.

Definition 2.
Let $\langle S, \delta \rangle$ be a metric space and let $f : S \mapsto S$ be a mapping from S to itself. The f is a *contraction mapping* (or simply *contraction*) on $\langle S, \delta \rangle$ if and only if:

$$\exists_{\epsilon \in [0,1)} \forall_{x,y \in S} \delta(f(x), f(y)) \leq \epsilon * \delta(x,y) \tag{1}$$

In other words, there is a constant $\epsilon \in [0,1)$ that the distance, in metric δ, of two elements of S transformed by f is less than or equal to the distance of those elements multiplied by ϵ.

Definition 3.
A metric space $\langle S, \delta \rangle$ is *complete*, if every Cauchy sequence of points in S has a limit that is also in S.

Theorem 1. (Banach fixed point theorem)
Let $\langle S, \delta \rangle$ be a non-empty complete metric space. Also, let $f : S \mapsto S$ be a contraction mapping on S. Then:

- *The map f admits one and only one fixed point $x^* \in S$ ($f(x^*) = x^*$).*
- *For arbitrary element $x_0 \in S$ the sequence (x_0, $f(x_0)$, $f(f(x_0))$, ...) converges and its limit is x^*:*

$$\lim_{i \to \infty} f^i(x_0) = x^*$$

4. Convergence Analysis of Quantum-Inspired Evolutionary Algorithm

To perform the convergence analysis of the considered class of algorithms all assumptions of the Banach fixed point theorem for the algorithm will be discussed.

1. Metric space S of all possible quantum populations is required.
2. The S needs to be a complete space.
3. A composition of $n \in \mathbb{N}^+$ consecutive iterations of the algorithm needs to create a mapping $f : S \mapsto S$.
4. The transformation f needs to be a contraction.

In the following subsections we will discuss all above points for the quantum evolutionary algorithm. The population metric space and the transformation of populations in consecutive generations will be considered.

Let us assume that the convergence analysis will be performed for the maximization of unimodal objective function $F : X \mapsto \mathbb{R}$. Also, let us denote by x^* the optimum solution of optimization problem:

$$x^* = \arg\max_{x \in X} F(x) \tag{2}$$

4.1. Metric space of quantum populations. Let S denote a set of all possible quantum populations Q of k elements. Also, let us assume that the representation of solutions has been settled, e.g. quantum individual is represented as single binary quantum chromosome. Thus, a measure of distance δ between elements from S is required to create a metric space.

Naturally, the metric δ should be based on a quality measure of quantum populations Q, so that its growth, not necessarily monotonic, in consecutive generations would imply decrease of the distance between elements of $\langle S, \delta \rangle$ space (convergence condition). Consequently, the two measures have to be chosen properly: a measure $eval^*$ of quantum genotypes quality as well as measure $Eval^*$ for quantum populations Q. The $Eval^*$ needs to be a deterministic measure. Otherwise, the distance δ, which needs to be based on $Eval^*$, would not met the definition of metric, stated in section 3.

Let us consider the following measure of quality of quantum genotypes $eval^*(q)$. Let $eval^* : Q \mapsto [0,1]$ be a probability of sampling the optimum value x^* with respect to probability distribution $Pr_q : \mathcal{F}_\Omega \mapsto [0,1]$ encoded in $q \in Q$:

$$eval^*(q) = Pr_q\left(\left\{\arg\max_{x \in X} F(x)\right\}\right) \tag{3}$$

It is important to notice that the proposed measure reaches its optimum for the quantum genotype q^* with no uncertainty involved, i.e. selecting exact point x^* from the X space ($Pr_{q^*}(\{x^*\}) = 1$).

Also, let us define $Eval^* : Q \mapsto [0,1]$ as an average value of $eval^*$ over the quantum population Q:

$$Eval^*(Q) = \frac{1}{|Q|} \sum_{q \in Q} eval^*(q) \tag{4}$$

Next, by analogy to (Michalewicz, 1996), let us define a distance between quantum populations $Q_1, Q_2 \in S$ as:

$$\delta(Q_1, Q_2) = \begin{cases} 0 \quad \text{if } Q_1 = Q_2 \\ |2 - Eval^*(Q_1)| + |2 - Eval^*(Q_2)| \\ \text{if } Q_1 \neq Q_2 \end{cases} \tag{5}$$

The distance $\delta(Q_1, Q_2)$ satisfies conditions for a metric given in section 3. Indeed, based on the definition of $\delta(Q_1, Q_2)$:

- $\delta(Q_1, Q_2) \geq 0$ for every $Q_1, Q_2 \in S$ and $\delta(Q_1, Q_2) = 0 \iff Q_1 = Q_2$
- $\delta(Q_1, Q_2) = \delta(Q_2, Q_1)$
- $\delta(Q_1, Q_2) + \delta(Q_2, Q_3) = |2 - Eval^*(Q_1)| + |2 - Eval^*(Q_2)| + |2 - Eval^*(Q_2)| + |2 - Eval^*(Q_3)| \geq |2 - Eval^*(Q_1)| + |2 - Eval^*(Q_3)| = \delta(Q_1, Q_3)$

Therefore, the $\langle S, \delta \rangle$ constitutes a metric space.

4.2. Completeness of $\langle S, \delta \rangle$ space. Selected metric space needs to be complete. In other words, every convergent sequence of elements from the $\langle S, \delta \rangle$ space needs to have a limit which belongs to the space. In a computer implementation of the algorithm this condition is always satisfied, as in computer memory the population belongs to a discrete, countable and finite space.

4.3. Transformation in the $\langle S, \delta \rangle$ space. Let us consider a transformation $f : S \mapsto S$ as a sequence of $n \in \mathbb{N}^+$ consecutive iterations with the property that the quality $Eval^*(Q(t+n))$ has improved compared to $Eval^*(Q(t))$. Thus, the proof of convergence requires the condition that the algorithm improves the probability of sampling the optimum solution x^*.

Contrary to the chapter in (Michalewicz, 1996) concerning convergence of „contraction mapping genetic algorithm”, in our case iteration number t is not incremented only when the improvement occurs. Instead, the transformation f is defined as the composition of certain, properly chosen, number of consecutive iterations with respect to the property that the improvement occurred. Consequently, no modifications to the general scheme of quantum evolutionary algorithm need to be introduced.

4.4. Transformation f as a contraction. Now, let us prove that if the transformation f improves quality $Eval^*$ of population, then f is a contraction mapping:

$$\exists_{\epsilon \in [0,1)} \forall_{Q_1, Q_2 \in S} \delta(f(Q_1), f(Q_1)) \leq \epsilon * \delta(Q_1, Q_2) \quad (6)$$

Indeed, based on the definition of transformation f:

$$Eval^*(Q1(t)) < Eval^*(f(Q1(t)) = Eval^*(Q1(t+n))$$

$$Eval^*(Q2(t)) < Eval^*(f(Q2(t)) = Eval^*(Q2(t+n))$$

Therefore:

$$\begin{aligned} & \boldsymbol{\delta(f(Q1(t)), f(Q2(t)))} \\ = \; & \delta(Q1(t+n), Q2(t+n)) \\ = \; & |2 - Eval^*(Q1(t+n))| + |2 - Eval^*(Q2(t+n))| \\ < \; & |2 - Eval^*(Q1(t))| + |2 - Eval^*(Q2(t))| \\ = \; & \boldsymbol{\delta(Q1(t), Q2(t))} \end{aligned} \quad (7)$$

Moreover, in a computer implementation of the algorithm the value of constant $\epsilon \in [0, 1)$ can be introduced into the right side of (7) and exactly calculated. Namely, the improvement of $Eval^*$ after the f transformation is greater than or equal to the precision of computer representation of numbers. Consequently, (6) is satisfied and f is a contraction mapping on S.

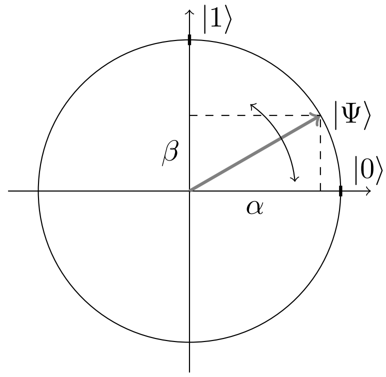

Fig. 2. Geometric illustration of binary quantum gene, represented as qubit with its imaginary part omitted. Along with increase of the angle between the vector $|\Psi\rangle$ and the horizontal axis, the probability of observing value 1 grows.

4.5. Convergence conditions. The above analysis enables to identify sufficient condition for convergence of the algorithm with respect to the Banach fixed point theorem. The algorithm converges in the complete metric space $\langle S, \delta \rangle$, if improvement of considered quality measure $Eval^*$ of quantum populations occurs in consecutive steps.

Corollary 1. *On the considered assumptions, quantum inspired evolutionary algorithm converges, if the probability of sampling the optimum element x^* grows as generation number t approaches infinity.*

If the condition from the corollary is satisfied, the quality measure $Eval^*$, that describes the average probability of sampling the optimum solution, grows inevitably.

5. Convergence of *Quantum-Inspired Genetic Algorithm*

Let us consider the Quantum-Inspired Genetic Algorithm (Han and Kim, 2000), which corresponds to the general evolutionary algorithm scheme, presented in section 2.

The algorithm uses a novel representation of gene that is based on the concepts of qubits. The qubit is a basic unit of quantum information. It is a normalised vector in a two dimensional vector space spanned by base vectors $|0\rangle$ and $|1\rangle$:

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad (8)$$

where $\alpha, \beta \in \mathbb{C}$, $|0\rangle = [1 \;\; 0]^T$, $|1\rangle = [0 \;\; 1]^T$ and $|\alpha|^2 + |\beta|^2 = 1$.

Observation of the qubit $|\Psi\rangle$ yields a value 0 with probability $|\alpha|^2$ and value 1 with probability $|\beta|^2$. The Fig. 2 illustrates a state of binary quantum gene, represented as qubit.

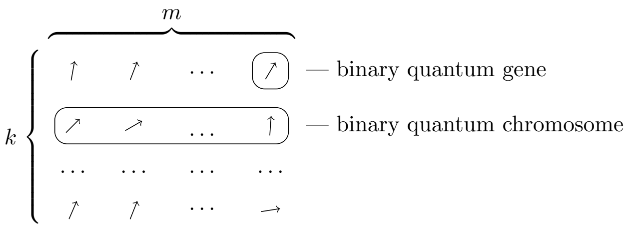

Fig. 3. Illustration of quantum population of k binary quantum chromosomes, each of length m.

The algorithm uses binary quantum chromosomes for representation of solutions, encoded as:

$$q_t = \left[\begin{array}{c|c|c|c} \alpha_1^t & \alpha_2^t & \dots & \alpha_m^t \\ \beta_1^t & \beta_2^t & \dots & \beta_m^t \end{array} \right] \quad (9)$$

Thus, a state of a quantum population in the generation t is depicted in Fig. 3. Each row in figure represents a state of binary quantum chromosome (numbered 1 to k), while each element in a row represents a state of binary quantum gene. For example, binary quantum chromosome denoted by $\rightarrow\uparrow\rightarrow\uparrow$, corresponds to the binary string 0101, while $\nearrow\nearrow\nearrow\nearrow$ corresponds to the binary quantum chromosome, which selects one of 2^4 binary strings: 0000, 0001, ..., 1111.

The pseudocode of Quantum-Inspired Genetic Algorithm, proposed by Han and Kim, has been presented in Fig. 5.

procedure QIGA
begin
$t \leftarrow 0$
initialize $Q(0)$
make $P(0)$ by observing $Q(0)$
evaluate $P(0)$
store the best solution among $P(0)$
while not termination-criterion **do**
  $t \leftarrow t + 1$
  make $P(t)$ by observing $Q(t-1)$ states
  evaluate $P(t)$
  update $Q(t)$ using quantum gates $U(t)$
  store the best solution among $P(t)$
end while
end

Fig. 4. Quantum-inspired Genetic Algorithm (Han and Kim, 2000)

In the beginning, the genes of all individuals are initialized with $\frac{\sqrt{2}}{2}|0\rangle + \frac{\sqrt{2}}{2}|1\rangle$ states, which corresponds to initial angles 45° between the state vectors and the base vectors $|0\rangle$, $|1\rangle$. Thus, initially, every quantum individual can be depicted as a sequence of vectors: $\nearrow\nearrow\dots\nearrow$. This is equivalent to uniform distribution of sampling the solution space.

The evaluation of individuals' fitness in the algorithm is performed by observation of Q states yielding P. In this step the search space is sampled with respect to probability distributions encoded by quantum individuals $Q(t)$. Thus, the evaluation is based on the value of objective function F in the randomly chosen point of the search space.

After sampling the space the algorithm stores the binary string of the best individual, if it outperforms the best solution that has been found in previous generations. Then, quantum rotation gates $U(\theta)$ are used to modify the probability distributions encoded in the population. The genes in quantum chromosomes are updated by the quantum rotation gates, according to the following principle of quantum computing:

$$g_{t+1} = U(\theta)g_t = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad (10)$$

where: $g_t = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ denotes the value of quantum gene in generation t and g_{t+1} denotes the value of quantum gene in generation $t+1$. The rotation direction for each gene of quantum chromosome depends, respectively, on the elements of binary string of the best solution found. For example, if the binary string 101 stores the best solution found, the first genes of quantum chromosomes are rotated toward $|1\rangle$ (positive rotation angle), the second genes are rotated toward $|0\rangle$ (negative rotation angle), and the third genes are rotated toward $|1\rangle$.

Consequently, the probability of sampling the neighbourhood of the best individual found so far increases in consecutive generations of the algorithm. When a better

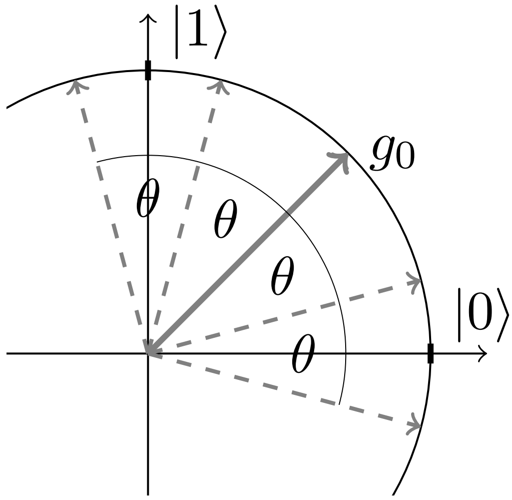

Fig. 5. The update operation based on the rotation angle θ that divides the initial angle (45°) with a non-zero remainder is incapable of creating a state of quantum gene that has no uncertainty.

solution is found, it replaces the best individual that has been found in previous generations.

Based on the assumptions of infinite number of generations and optimization of unimodal objective function F, the considered algorithm converges to the optimum solution, if genetic operators can not create such population Q that eliminates probability of sampling the optimum solution x^*. The probability is not eliminated, if proper rotation angles θ are selected. Therefore, the algorithm converges for such rotation angles that are incapable of creating a population with no uncertainty prematurely.

Since in the beginning of the algorithm all quantum genes are initialized with linear superpositions $\frac{\sqrt{2}}{2}|0\rangle + \frac{\sqrt{2}}{2}|1\rangle$, which corresponds to initial angles 45°, by this line of reasoning, the algorithm converges, if rotation angles divide the initial angle 45° with a non-zero remainder which has been demonstrated in Fig. 5. In this situation the algorithm never lose the probability of sampling the optimal solution x^*. Also, taking into consideration that the algorithm modifies the probability distributions according to this best solution found, the probability of sampling the neighbourhood of this solution grows inevitably. Consequently, the probability of sampling the optimum element x^* grows as generation number t approaches infinity, which satisfies the convergence condition given in section 4.5.

6. Conclusions

In this paper a general Quantum-Inspired Evolutionary Algorithm has been considered and necessary conditions for convergence of the algorithm have been discussed based on the Banach fixed point theorem. The analysis has been demonstrated by the Quantum-Inspired Genetic Algorithm.

The new measure of quality for quantum genotypes has been proposed and the algorithm has been considered in the relevant metric space. The metric space allowed to identify conditions for convergence of quantum evolutionary algorithms. It has been presented that the additional element of randomness in evaluation of individuals' fitness has no influence on the convergence of the algorithm.

In the general case there is no guarantee of convergence to the global optimum. For more than one optimum solution the algorithm does not met necessary condition of contraction mapping, as for different optimum solutions $\delta(f(Q_1), f(Q_2)) = \delta(Q_1, Q_2)$.

Also, it is important to notice that evolutionary algorithms are convergent (or not), regardless of selected metric space. Convergence in a computer implementation of the algorithm means that, starting at some generation, populations are equal to each other, therefore their distance equals 0, regardless of selected measure.

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Robert Nowotniak received the MSc degree in computer science in 2008 from the Institute of Computer Science of the Technical University of Łódź, Poland. Currently, he is a PhD student at the Computer Engineering Department. His main research interests include evolutionary computing, quantum computing and numerical optimization.

Jacek Kucharski graduated from the Technical University of Łódź (TUL) in 1990. He received the Ph.D. and D.Sc. degrees in 1993 and 2004 respectively from the same University. Currently, he is appointed at the position of Professor at the Computer Engineering Department, TUL. His research interest focuses mainly on artificial intelligence techniques and their application in various areas.

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